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LETTER TO THE EDITOR

Geometrical properties of matrix solutions of the nonlinear Klein–Gordon equation

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Abstract. We have constructed some matrix solutions of a nonlinear Klein–Gordon equation and proposed a relation between these solutions and $SU(n)$ matrix groups. We have also established the correspondence between the solutions and the rotations around fixed vectors whose endpoints form an octahedron.

In the previous papers [1, 2] the solutions of the Klein–Gordon (KG) equation were given as complex and hypercomplex ones. Here we present a uniform definition of matrix solutions u_n of the nonlinear KG equation. Such solutions are constructed on the basis of the unitary anti-Hermitian anticommuting $n \times n$-matrices. It is shown that solution u_1 draws a helical curve, solution u_2 realizes a rotation of a unit sphere, while solution u_3 realizes a rotation around fixed vectors whose endpoints form an octahedron.

Consider the KG equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\mathrm{d}Q}{\mathrm{d}u} = 0 \qquad Q(u) = \frac{\lambda^2}{4}(u^2 - 1)^2.$$

The case of the potential $V(\psi)$ in [3, p 189] can be reduced to this one by changing the variable $\psi = um/\lambda$ and parameter λ. To simplify the mathematics we choose the direction $x = \sum_{j=1}^{3} c_j x_j$ where $\sum_{j=1}^{3} c_j^2 = 1$ and then pass to the moving frame of reference $z = x - vt$ where v is the velocity.

Similar to [1, 2], we construct matrix solutions of the KG equation as

$$u_n(\alpha z) = -\tanh(\alpha z)E_n + \mathrm{sech}(\alpha z)\sum_{j=1}^{m} a_j M_j \qquad \sum_{j=1}^{m} a_j^2 = 1, \quad \alpha = \lambda\sqrt{\frac{2}{1 - v^2}}$$

where $v^2 < 1$, $n = 1, 2, \ldots$, and E_n is the unit $n \times n$-matrix. The complex linear-independent $n \times n$-matrices $M_j (j = 1, 2, \ldots, m)$ should possess the following properties: they are unitary ($M_j^* = M_j^{-1}$), anti-Hermitian ($M_j^* = -M_j$), and anticommuting ($M_i M_j = -M_j M_i$). The symbol $*$ denotes the transition to a complex conjugate transposed matrix. For $n = 1$ we should set $m = 1$ and $M_1 = \mathrm{i}$. The fundamental property of the solutions is

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\alpha}{2}(u^2 - 1) \qquad \text{for} \quad u = u_n(\alpha z).$$

Now we define the function $\phi \equiv \phi(\alpha z) = \mathrm{arccot}(\sinh(-\alpha z))$ and write the accompanying equalities as

$$\cos(\phi) = \tanh(-\alpha z) \qquad \sin(\phi) = \mathrm{sech}(\alpha z).$$

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Note that function ϕ increases from 0 to π if z varies from $-\infty$ to ∞. For $\mathrm{i}A_j = M_j$ and $\boldsymbol{a}\boldsymbol{A} = \sum_{j=1}^{m} a_j A_j$ where A_j is a Hermitian matrix in contrast to M_j, we rewrite the solution u_n in the form

$$u_n(z) = \cos(\phi)E_n + \mathrm{i}\sin(\phi)\boldsymbol{a}\boldsymbol{A} = \exp(\mathrm{i}\phi\boldsymbol{a}\boldsymbol{A}).$$

For fixed $n = 2, 3, \ldots$ and $\boldsymbol{a} = (1, 0, \ldots, 0)$ this expression establishes the one-to-one correspondence between solutions u_n and unitary Hermitian $n \times n$-matrices A_1. Moreover, if the expansion $\boldsymbol{a}\boldsymbol{A}$ belongs to a space of unitary unimodular ($\det(\boldsymbol{a}\boldsymbol{A}) = 1$) matrices, then this expression gives a relation between solutions u_n and $SU(n)$ matrix groups.

Consider, next, the case $n = 2$. As is known [1, 2] in this case $m = 3$, $M_1 = \mathrm{i}\sigma_1$, $M_2 = -\mathrm{i}\sigma_2$, and $M_3 = \mathrm{i}\sigma_3$. Hence

$$u_2(\alpha z) = \exp(\mathrm{i}\phi\boldsymbol{a}\sigma) \qquad \sigma = (\sigma_1, -\sigma_2, \sigma_3)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are unitary Hermitian and anticommuting Pauli matrices. The sign minus before σ_2 is justified by the rule of right-hand rotation. The unitary unimodular matrix u_2 belongs to the $SU(2)$ group and, as is proved in [4, p 41], such a matrix realizes rotation of the unit sphere around vector $\boldsymbol{a}$ by angle 2ϕ.

The solution $u_1 = \exp(\mathrm{i}\phi)$ can be represented in the complex space $(\mathrm{Re}\, u, \mathrm{Im}\, u, z)$ as a helical curve with axis z. The solution u_2 realizes a rotation of the unit sphere (or a vector field) with a centre at a moving point on the helical curve u_1. Depending on the choice of vector $\boldsymbol{a}$ we can obtain different solutions u_2. Moreover, the function u_2 shifted by angle $\pi/2$ also presents the solution $\tilde{u}_2 = \exp(\mathrm{i}(\phi + \pi/2)\boldsymbol{a}\sigma)$ of the KG equation with potential $Q = \lambda^2(\tilde{u}^2 + 1)^2/4$ and $\alpha^2 = 2\lambda^2/(v^2 - 1)$ where $v^2 > 1$. Thus we can construct solutions similar to the expressions of the vector fields W^0, Z^0, $W^{\pm}$ in the Glashow–Salam–Weinberg theory. Indeed, if $\boldsymbol{a} = (0, 0, 1)$, then for ϕ and $\phi + \pi/2$, respectively, we find

$$W^0 = \cos(\phi)E_2 + \sin(\phi)M_3 \qquad Z^0 = -\sin(\phi)E_2 + \cos(\phi)M_3.$$

In the case $\boldsymbol{a} = (1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$ the solution u_2 is associated with the $W^{\pm}$.

We can now proceed to the case $n = 3$. As is proved in [2], one cannot find two unitary anti-Hermitian 3×3-matrices that anticommute with each other. Therefore $m = 1$ and for $M_1 = \mathrm{i}A$ we have

$$u_3(\alpha z) = \cos(\phi)E_3 + \sin(\phi)M_1 = \exp(\mathrm{i}\phi A).$$

Now we construct a basis in a space of unitary Hermitian 3×3-matrices as

$$\mu_j = \begin{pmatrix} \hat{\sigma}_j & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \mu_{j+6} = \begin{pmatrix} -1 & 0 \\ 0 & \hat{\sigma}_j \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3$$

where $\hat{\sigma}_2 = -\sigma_2$ and $\hat{\sigma}_1, \hat{\sigma}_3$ are equal to σ_1, σ_3, respectively.

$$\mu_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mu_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & -1 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad \mu_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\mu_3 + \mu_6 + \mu_9 = -E_3$. The matrices μ_j for $j = 4, 5, 6$ are obtainable from the first triple by cyclic rearrangement of the lines and columns as $(1, 2, 3) \to (2, 3, 1)$; and the third triple μ_j for $j = 7, 8, 9$ is obtained from the second one by rearranging cyclically the lines and columns as $(2, 3, 1) \to (3, 1, 2)$.

It is easy to verify that the matrices μ_j for $j = 1, \ldots, 9$ are linear-independent, unitary, Hermitian, and unimodular. Any matrix A from $SU(3)$ can be represented as an expansion of matrices μ_j. Consider a more instructive case when

$$A = \sum_{j=1}^{3} c_j \mu_j = \begin{pmatrix} c_3 & c_1 + \mathrm{i}c_2 & 0 \\ c_1 - \mathrm{i}c_2 & -c_3 & 0 \\ 0 & 0 & -c_1 - c_2 - c_3 \end{pmatrix}.$$

The unitarity and the property $\det(A) = 1$ lead to the equalities

$$\sum_{j=1}^{3} c_j^2 = 1 \qquad \text{and} \qquad \sum_{j=1}^{3} c_j = 1.$$

To express the coefficients c_j as functions of the angular parameter θ we set $c_3 = a$ and $c_1 + \mathrm{i}c_2 = b\exp(\mathrm{i}\theta)$. Then after some calculations we obtain the formulae

$$c_1 = 1 - (1 + c_3)\sin^2(\theta)$$
$$c_2 = 1 - (1 + c_3)\cos^2(\theta)$$
$$c_3 = (1 + \cos(\theta)\sin(\theta))^{-1} - 1.$$

Let us consider the extreme points of the function $c_3(\theta)$. At $\theta = \pi/4$ or $5\pi/4$ one has $c_3 = -\frac{1}{3}$ (minimum), $c_1 = c_2 = \frac{2}{3}$; at $\theta = 3\pi/4$ or $7\pi/4$ one has $c_3 = 1$ (maximum), $c_1 = c_2 = 0$. It is clear that the formulae for c_j do not vary if we replace μ_j by μ_{j+3} or μ_{j+6} in the expansion of A. Thus we can investigate the following three independent solutions

$$u_{3r} = \exp(\mathrm{i}\phi c_j \mu_j) \qquad u_{3y} = \exp(\mathrm{i}\phi c_j \mu_{j+3}) \qquad u_{3b} = \exp(\mathrm{i}\phi c_j \mu_{j+6})$$

where two equal indexes j are meant to be summed over values $j = 1, 2, 3$.

To understand what kind of a rotation is realized by these solutions, we start with such an argument. If the matrix u_{3r} as an operator were applied to the vector (q_1, q_2, q_3) where $q_3 = 0$, then this action would be equivalent to an action of the 2×2-matrix $\exp(\mathrm{i}\phi \boldsymbol{c\sigma})$ on the doublet (q_1, q_2). This reduction to the doublet is analogous to that considered in [5, p 216]. Moreover, taking into account the rearrangement in the construction of μ_{j+3} and μ_{j+6}, the matrices u_{3y} and u_{3b} should be applied to the vectors (q_2, q_3, q_1) and (q_3, q_1, q_2), respectively. This means that matrices u_{3r}, u_{3y}, u_{3b} act in local frames of reference (e_1, e_2, e_3), (e_1', e_2', e_3'), and (e_1'', e_2'', e_3''), respectively, with the common origin $(q_3 = 0)$ provided that $(e_1, e_2, e_3) = (e_2', e_3', e_1') = (e_3'', e_1'', e_2'')$. Thus we state that each matrix realizes rotation around the vector (c_1, c_2, c_3) in its own frame of reference; in other words, these matrices realize the rotations around the vectors (c_1, c_2, c_3), (c_2, c_3, c_1), and (c_3, c_1, c_2) in the original frame of reference (e_1, e_2, e_3).

For a more interesting case when $c_3 = -\frac{1}{3}$ the endpoints

$$(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}) \qquad (\tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{2}{3}) \qquad (-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3})$$

of the triple of vectors form an equilateral triangle. The opposite points (on the unit sphere) form an analogous triangle. All the six points form an octahedron with a sidelength of $\sqrt{2}$. Among other things, the doublet (q_1, q_2) mentioned above can be formed, for example, from the endpoints of the vectors (c_1, c_2, c_3) and $(-c_1, -c_2, -c_3)$.

Note that solutions u_n constructed in [6] for the KG equation with the special potential

$$Q_k(u) = \frac{\lambda^2}{4}(u^2)^{\frac{k-1}{k}}((u^2)^{\frac{1}{k}} - 1)^2 \qquad k = 1, 2, \ldots$$

can be represented in the form of $\exp(\mathrm{i}k\phi \boldsymbol{aA})$ and hence can be considered in the similar manner.

We hope that the construction of the solutions u_3 will be useful in the particle physics.

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